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Efficiency dynamics on scale-free networks with tunable degree exponent

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Abstract. A model, introduced earlier for the dynamics of a generic efficiency measure in a population of agents by Majumdar and Krapivsky (Phys. Rev. E **63**, 054101 (2001)), is investigated on scale-free networks whose degree distribution follows a power law with the tunable exponent γ. The model shows a delocalization transition from a stagnant phase to a growing one when decreasing the degree exponent γ of scale-free networks. By taking into account the specific dynamical properties of the model and the geometrical properties of scale-free networks, we predict the appearance of this critical transition. This work is useful for understanding these kinds of transitions occurring in many dynamical processes on scale-free networks.

PACS. 89.75.Hc Networks and genealogical trees – 05.10.-a Computational methods in statistical physics and nonlinear dynamics – 87.23.Ge Dynamics of social systems

1 Introduction

Complex systems consist of many constituents such as individuals, substrates, and companies in social, biological, and economic systems, etc. They show cooperative phenomena between constituents through diverse interactions and adaptations to the pattern created by them [1,2]. Interactions may be described in terms of complex networks, consisting of vertices and edges, where vertices (edges) represent the constituents (their interactions). Various models have been developed in order to describe the structure and properties of these networks [3–8]. Among these models, the small-world network introduced by Watts and Strogatz attracts a great deal of attention [6,7]. A great number of dynamics of social processes were studied on the small-world network, such as disease spreading, formation of public opinion, distribution of wealth, etc. [9–14]. In the past, we developed a simple model that describes the dynamics of efficiencies of competing agents [15] on a small-world network [16]. In this model communications among agents lead to the increase of efficiencies of underachievers, and the efficiency of each agent can increase or decrease irrespective of other agents. The model can also be considered as a polynuclear growth model with desorption, where the degrees of freedom are the heights of a growing interface [17–19].

However, several studies on real complex networks from different fields as biology, economy, or sociology have shown that the degree of nodes (number of edges connected to each node) follows a scale-free power-law distribution like $P(k) \sim k^{-r}$, where $P(k)$ denotes the frequency of the nodes that are connected to k other nodes. Such networks, called scale-free networks, are ubiquitous in nature, such as the World Wide Web [20–22], the Internet [23,24], the citation network [25], the author collaboration network of scientific papers [26–28], and the metabolic networks in biological organisms [29]. Thus, a lot of dynamic models have been proposed based on the scale-free network, such as disease spreading [30], sandpile model [31], Bak-Sneppen model [32], and load distribution model [33].

In this paper we develop the simple efficiency model on scale-free networks with the tunable degree exponent γ. The model shows a delocalization transition from a stagnant phase to a growing one when decreasing the degree exponent γ of scale-free networks instead of increasing the disorder of the small-world networks [16]. The present work will be useful for understanding the critical transition appearing in many dynamical processes on scale-free networks with tunable degree exponent γ.

2 Model and method

2.1 Scale-free networks

A scale-free network can be generated by using the static model [31,32]. The network consists of N nodes. Each

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node is indexed by an integer i ($i = 1, \ldots, N$) and is assigned a weight equal to $w_i = i^{-\alpha}$. Here α is a control parameter in [0,1) and is related to the degree exponent via the relation $\gamma = 1 + 1/\alpha$ for large N. Adjusting the control parameter α in [0,1) we can get various exponents γ in the interval (2,∞). At each step, we randomly select two different nodes i and j with probabilities equal to the normalized weights, $w_i/\Sigma_k w_k$ and $w_j/\Sigma_k w_k$, respectively, and attach an edge between them unless one exists already. This process is repeated until the mean degree of the network becomes $2m$. In this work, we take $N = 10^4$ and $m = 5$.

2.2 Efficiency model

In this work, the evolution of the efficiencies has analogy to that used in references [15] and [16], which may mimic the dynamics of efficiencies of competing agents such as airlines, travel agencies, insurance companies, etc. The efficiency of each agent is expressed as a single nonnegative number. The efficiency of every agent can, independent of other agents, increase or decrease stochastically by a certain amount which we set equal to unity. In our model, the interactions between the agents of the population are expressed by a scale-free network.

Our efficiency model on the scale-free network can be described as follows. Each agent is represented by an node in the network. The agent i is characterized by a nonnegative integer number $h_i(t)$, which stands for its efficient level. It has been verified that the calculated results not depend on the initial conditions for present model. Thus, we choose the efficiency of each agent $h_i(0) = 0$ as the initial conditions for simplicity. We study the evolution of the efficiencies of N agents in the scale-free network by Monte Carlo (MC) simulations. At MC step t, an agent is selected randomly (say, it is the agent i) and its efficient $h_i(t)$ is changed from $h_i(t)$ to one of three values with corresponding probability as follows:

(i) $h_i(t)$ is replaced with the value of $\max[h_i(t), h_j(t)]$ with probability $1/(1 + p + q)$, where the agent j is one of the agents which are linked to the agent i. It is based on the fact that each agent tries to equal its efficiency to that of a better performing agent in order to stay competitive.

(ii) $h_i(t)$is changed into $h_i(t) + 1$ with probability $p/(1 + p + q)$. This is in agreement with the fact that each agent can increase its efficiency, say due to innovations, irrespective of other agents.

(iii) $h_i(t)$ is substituted by $h_i(t) - 1$ with probability $q/(1 + p + q)$. This is consistent with to the fact that each agent can lose its efficiency due to unforeseen problems such as labor strikes. Note, however, that since $h_i(t) \geq 0$, this move can occur only when $h_i(t) \geq 1$.

The renewal of efficiency continues step by step. After each MC step the 'time' is increased by $1/N$. Thus after 1 time step on the average all agents in the network have made an update. In order to investigate the effect of the topological structure of the population on the dynamics of efficiencies, the value of the parameters p and q are fixed in the present model. Without loss of generality, we choose $p = 1$ and $q = 18$, where the mean-field theory predicts a stagnant phase of efficiencies [15].

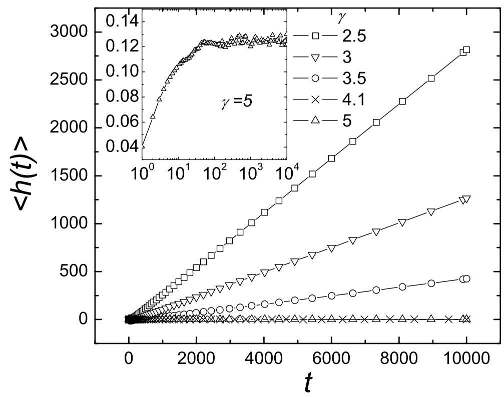

Fig. 1. The average efficiency $\langle h(t)\rangle$ as a function of time t for a set of degree exponent γ. The size of the system $N = 10^4$.

3 Results and discussion

To investigate the dynamics of efficiency on the scale-free networks, extensive numerical simulations have been completed. In the simulations, we take the size of the system $N = 10^4$ and the degree exponent γ in [2,∞). To reduce the effect of fluctuation on calculated results, for every system with size N, the calculated results are averaged over both 10 different network realizations and 10 independent runs for each network realization.

Firstly, we have calculated the average efficiency $\langle h(t)\rangle$ per agent as a function of time t for different degree exponent γ. Figure 1 shows the average efficiency $\langle h(t)\rangle$ as a function of time t for the system with size $N = 10^4$ for a set of different degree exponent γ. It can be seen from Figure 1 that there exists a critical degree exponent γ_c. When the degree exponent $\gamma < \gamma_c$, the average efficiency $\langle h(t)\rangle$ per agent will linearly increase with time t (e.g. $\gamma = 2.5$), When the degree exponent $\gamma > \gamma_c$, the average efficiency $\langle h(t)\rangle$ per agent approaches a low constant in the long time (e.g. $\gamma = 5$). This indicates that there may exist a critical phase transition from a growing phase of efficiency to a stagnant phase of efficiency at a critical degree exponent γ_c. To dwell on this transiton, the growth rate v of the average efficiency $\langle h(t)\rangle$ per agent in the long time limit is expressed as

$$v \equiv \frac{d\langle h(t)\rangle}{dt} = \frac{d\frac{1}{N}\sum_{i=1}^{N} h_i(t)}{dt}. \quad (1)$$

We calculate the growth rate v as a function of the degree exponent γ for several systems with size $N = 10^4$. The results are shown in Figure 2a. The results show that as for small γ, the growth rate $v > 0$ and for large γ, $v \to 0$ in the long time limit. So there exists a transition at a certain value γ_c. As $\gamma > \gamma_c$, the growth rate v is equal

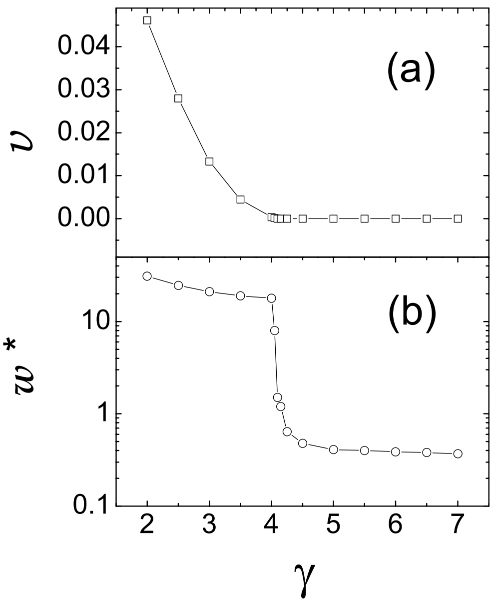

Fig. 2. The growth rate v of the average efficiency (a) and the asymptotic efficiency fluctuation w^* (b) as a function of the degree exponent γ of the network. The size of the system $N = 10^4$.

to zero; as $\gamma < \gamma_c$, v increases rapidly with γ. As $\gamma \approx \gamma_c$, the growth rate v transits from zero to a finite value, which corresponds to the transition of the system from a stagnant phase to a growing one. To further research the transition, we calculate the efficiency fluctuation $w(t)$ of the system, where the efficiency fluctuation $w(t)$is defined as [16]

$$w^2(t) = \frac{1}{N}\sum_{i=1}^{N}\left(h_i(t) - \langle h(t)\rangle\right)^2. \qquad (2)$$

The efficiency fluctuation $w(t)$ inclines to a constant $w^* = \langle w(t \to \infty)\rangle$ in the long-time limit. For several systems with size $N = 10^4$, the asymptotic value w^* as a function of the degree exponent γ are shown in Figure 2b. It can been seen from Figure 2b that when $\gamma > \gamma_c$, the fluctuation w^* tends to a small value of about 0.36; when $\gamma < \gamma_c$, the fluctuation w^* attains to the maximum close to 30; and when $\gamma \approx \gamma_c$, the fluctuation w^* sharply jumps from the small value to the maximum one. The results also prove that there exists a transition at a certain intermediate value of γ. For present system with $N = 10^4$, we obtain $\gamma_c = 4.05$, which corresponds to the inflexion of the curves in Figure 2b.

To understand the critical behavior, we analyze the dynamical properties of present model. In the model, three facts make contributes to the time evolution of $\langle h(t)\rangle$: (i) learning from its linked agents leads to the increase of $\langle h(t)\rangle$; (ii) innovation causes the increase; (iii) unforeseen problems bring about the decrease. Therefore, the growth rate v can be expressed as [15,16]

$$v(t) \equiv \frac{d\langle h(t)\rangle}{dt} = \frac{Aw(t) + p - qs(t)}{1 + p + q}, \qquad (3)$$

where A is a proportional factor depending on the degree exponent γ, and $s(t)$ is the probability of an agent having a nonzero efficiency. On the right-hand side of equation (3), The first term represents the increase due to learning from its linked agents, which is proportional to the nonuniform degree w^* of efficiencies among agents. The second term quantifies the increase caused by innovation. The last term indicates the decrease resulted from unforeseen problems and the reduction only take place if the agent has a nonzero efficiency. Substituting the values of p and q in equation (3), we obtain

$$v(t) = \frac{1}{20}[1Aw(t) + 1 - 18s(t)]. \qquad (4)$$

It can be seen from equation (4) that there exists a critical transition at an intermediate degree exponent γ_c of the network. Cohen and Havilin have indicated that scale-free networks are ultrasmall. As γ decreases, the average distance d of the network becomes much small [34]. Thus, the communications among agents on scale-free networks increase as γ decreases. Also it has been indicated that there exists a critical transition at an intermediate degree exponent γ_c of the network. As γ decreases, the network inhomogeneity increases. The average degree of the nearest neighbor of a node is $\kappa = \langle k^2\rangle/\langle k\rangle$, which can be approximated by [35]

$$\kappa = \langle k^2\rangle/\langle k\rangle = \left(\frac{\gamma-2}{\gamma-3}\right)\left(\frac{k_{cut}^{3-\gamma} - m^{3-\gamma}}{k_{cut}^{2-\gamma} - m^{2-\gamma}}\right), \qquad (5)$$

where, $k_{cut} = mN^{1/(\gamma-1)}$. The average distance of networks d can be expressed as [36]

$$d \approx \frac{\ln N}{\ln(\kappa - 1)}. \qquad (6)$$

It can be seen from equations (5) and (6) that the average distance d decreases as the network inhomogeneity increases. As the average distance decreases, communications among agents increase. So the network inhomogeneity enhances growth processes. Therefore, as $\gamma > \gamma_c$, the factor A is small because of weak communications among agents. The efficiencies of all agents are small, and the corresponding fluctuation $w(t)$ is also small. So, it can been expected that the first two terms and the last term on the right-hand side of the equation (4) will cancel each other and the probability with nonzero efficiency reaches the asymptotic time-independent value, i.e. $s = (Aw^* + 1)/18$ in the long time limit. This represents that the growth rate $v = 0$ and the average efficiency per agent $\langle h\rangle$ becomes a constant in the long time limit. We call this phase the "stagnant" phase [16]. However, as $\gamma < \gamma_c$, w and the proportional factor A are large due to strong communications among agents. After a long time, the fluctuation $w(t)$ attains to the stable value w^*, and the probability with nonzero efficiency reaches the maximum value of $s = 1$, but the last term on the right-hand side of equation (4) is still less than the sum of the fore two terms. So, the growth rate $v = (Aw^* - 17)/20$, which corresponds to that the

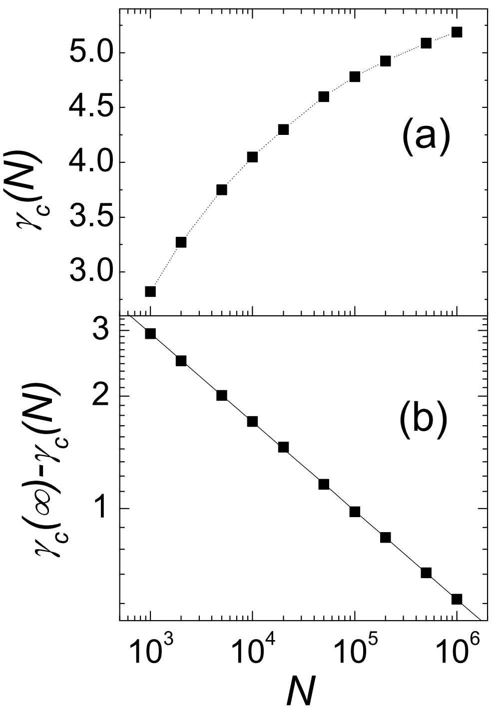

Fig. 3. The influence of the system size N on the critical degree exponent γ_c. (a) $\gamma_c(N)$ as a function of the system size N. (b) The deviation of the apparent critical value $\gamma_c(N)$ from the true critical value $\gamma_c(\infty)$ as a function of size N on a log-log plot, where $\gamma_c(\infty)$ is chosen to be 5.76. The symbols are the simulation results, and the straight line is the least-square fit to the data.

average efficiency per agent $\langle h(t)\rangle$ increases linearly with time t. We call this phase the "growing" phase [16]. It can been seen from equations (5) and (6) that the average distance d increases as γ increases. When $\gamma = \gamma_c$, we can get d_c from equations (5) and (6). When average distance takes the value of d_c, the communications among agents make the increase in efficiency per agent be equal to the loss in efficiency per agent.

In the following, we discuss the influence of the system size on the critical degree exponent γ_c. Figure 3a shows that the γ_c as a function of the system size N ranging from 10^3 to 10^6. It can be seen from Figure 3a that with the increase of the system size N the critical value $\gamma_c(N)$ increases and tends to a constant value, which corresponds to the true critical value $\gamma_c(\infty)$ for the infinite-size system. According to the finite-size effects of the systems, the apparent critical point $\gamma_c(N)$ and true critical point $\phi_c(\infty)$ are expected to scale with the size N as [37]

$$\gamma_c(\infty) - \gamma_c(N) \sim N^{-1/\nu}, \tag{7}$$

where ν is the critical shift exponent. To obtain the value of true critical point $\gamma_c(\infty)$ and critical exponent ν, Figure 3b plots the critical deviation $\gamma_c(\infty) - \gamma_c(N)$ as a function of the system size N on a log-log plot. When the true critical value is chosen to be $\gamma_c(\infty) \simeq 5.76$, we obtain the best power-law relation of the data by using equation (7). The excellent linear dependence in Figure 3b indicates that the finite-size scaling relation equation (7) well describes the present simulation results. From Figure 3b we also obtain the critical exponent $\nu \simeq 4.2$ by means of the least-square fit to the data.

Finally, we discuss the influence of the value of p and q chosen on the critical γ_c. We make simulations at a set of points in the (p, q) plane, where the mean-field theory predicts a stagnant phase of efficiencies [15]. By means of a finite-size scaling analysis, we obtain that both the critical γ_c and the exponent ν depend on the precise values of p and q chosen. For example, when $p = 1$ and $q = 20$, corresponding $\gamma_c \simeq 4.26$ and $\nu \simeq 2.7$.

4 Conclusions

On scale-free networks with the tunable degree exponent γ, the dynamics of efficiencies of competing agents shows a critical behavior. The results indicate that there exists a delocalization (or depinning) phase transition from a stagnant phase to a growing one at a critical degree exponent γ_c of the network. Above the critical point, $\gamma \geq \gamma_c$, the system is stagnant, the average efficiency per agent approaches a constant; below it, $\gamma < \gamma_c$, the average efficiency increases linearly with time. By taking into account the specific dynamical properties of the model and geometrical propertied of scale-free network, we predict the critical transition, which depends on the exponent of the degree distribution. This critical transition can also take place as the network becomes more and more skewed. This is a new feature of scale-free network, that of enhancing growth processes thanks to their inhomogeneity.

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